Symmetry and special relativity in Finsler spacetime with constant curvature

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Within the framework of projective geometry, we investigate kinematics and symmetry in (α, β) spacetime-one special types of Finsler spacetime. The projectively flat (α, β) spacetime with constant flag curvature is divided into four types. The symmetry in type A-Riemann spacetime with constant sectional curvature is just the one in de Sitter special relativity. The symmetry in type B-locally Minkowski spacetime is just the one in very special relativity. It is found that type C-Funk spacetime and type D-scaled Berwald's metric spacetime both possess the Lorentz group as its isometric group. The geodesic equation, algebra and dispersion relation in the (α, β) spacetime are given. The corresponding invariant special relativity in the four types of (α, β) spacetime contain two parameters-the speed of light and a geometrical parameter which may relate to the new physical scale. They all reduce to Einstein's special relativity while the geometrical parameter vanishes.

I. INTRODUCTION

Lorentz Invariance (LI) is one of the foundations of the Standard model of particle physics. Of course, it is very interesting to test the fate of the LI both on experiments and theories. The theoretical approach of investigating the LI violation is studying the possible spacetime symmetry, and erecting some counterparts of special relativity. Recently, there are a few counterparts of special relativity. The first one is doubly special relativity (DSR)[1–3]. In DSR, the Planck-scale effects have been taken into account by introducing an invariant Planckin parameter κ. Together with the speed of light c, DSR has two invariant parameters. The second one is very special relativity (VSR) [4, 5]. Coleman and Glashow have set up a perturbative framework for investigating possible departures of local quantum field theory from LI. The symmetry group of VSR is some certain subgroups of Poincare group, which contains the spacetime translations and proper subgroups of Lorentz transformations. The last is the de Sitter(dS)/anti de Sitter(AdS) invariant special relativity (dSSR) [7, 8]. The dSSR suggests that the principle of relativity should be generalized to constant curvature spacetime with radius R in Riemannian manifold.

In fact, the three kinds of modified special relativity share common ground. Historically, Snyder proposed a quantized spacetime model [9]. In his model, the spacetime coordinates were defined as translation generators of dS-algebra $\mathfrak{so}(1,4)$ and become noncommutative. It has already been pointed out in Ref. [10] that there is a dual one-to-one correspondence between Snyder's quantized spacetime model as a DSR and the dSSR. Actually, the Plackin parameter κ in DSR is related to the parameter a in Snyder's model in addition to c. Furthermore, the dSSR can be regarded as a spacetime counterpart of Snyder's model. VSR can be realized on a noncommutative Moyal plane with light-like noncommutativity [11]. Thus, the three kinds of modified special relativity all have noncommutative realization.

On the other hand, these counterparts of special relativity have connections with Finsler geometry [12], which is a natural generalization of Riemannian geometry. The noncommutativity effects may be regarded as the deviation of Finsler spacetime from Riemann spacetime. Ref.[13] gave a canonical description of DSR and showed that the DSR admits a modified dispersion relation (MDR) as well as noncommutative κ-Minkowskian phase space. Furthermore, Girelli *et al.*[14] showed that the MDR in DSR could be incorporated into the framework of Finsler geometry. As for VSR, Gibbons *et al.* have pointed out that general VSR is Finsler Geometry [15].

Therefore, It is reasonable to assume that these counterparts of special relativity may have a corporate origin in Finsler geommetry. In order to investigate the counterpart of special relativity in a systematic way, first, we should erect the inertial frames in Finsler spacetime. Second, we should investigate the symmetry in Finsler spacetime. The way of describing spacetime symmetry in a covariant language (the symmetry should not depend on any particular

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choice of coordinate system) involves the concept of isometric transformations. In fact, the symmetry of spacetime is described by the so called isometric group. The generators of isometric group is directly connected with the Killing vectors[16]. Actually, the symmetry of deformed relativity has been studied by investigating the Killing vectors[17]. It is well known that the isometric group is a Lie group in Riemannian manifold. This fact also holds in Finslerian manifold[18]. The counterparts of Poincare algebra in Finsler spacetime could be studied. At last, we should give the kinematic and dispersion law in Finsler spacetime.

This paper is organized as follows. In Sec.2, we present basic notations of Finsler geometry and discuss inertial frames in Finsler spacetime. In Sec.3, we use the isometric group to investigate the symmetry of Finsler spacetime. In Sec.4, we discuss the kinematics in projectively flat (α, β) spacetime with constant flag curvature. The isometric groups and the corresponded Lie algebras for different types of (α, β) spacetime are given. At last, we give the concluding remarks. The counterpart of sectional curvature in Riemann geometry-flag curvature is introduced in appendix.

II. FINSLER SPACETIME

The inertial frame means a particle in it continue at rest or in uniform straight motion. In an inertial system, the inertial motion is described by

$$x^i = v^i(t - t_0) + x_0^i, \quad v^i \equiv \frac{dx^i}{dt} = consts. \tag{1}$$

It should be notice that such definition for inertial motion (1) does not involve any specific requirements on the metric of spacetime. In fact, Einstein just assumed that the spacetime should be Euclidean which inherited from Newton[19]. If we loose the requirement that the spacetime should be Euclidean and require that the spacetime should be Riemannian, there exists three classes of inertial frames. Historically, de Sitter first used the projective coordinates (or Beltrami coordinates) to erect a spacetime with constant sectional curvature-the de Sitter spacetime. De Sitter used his dS spacetime to debate with Einstein on 'relative inertial'. Actually, the dS spacetime is one kinds of locally projectively flat spacetime.

A spacetime is said to be locally projectively flat if at every point, the geodesics are straight lines

$$x^\mu(\tau) = f(\tau)m^\mu + n^\mu, \tag{2}$$

where τ is the parameter of the curve, $f(\tau)$ is a function which depends on the metric of spacetime and m^μ, n^μ are constants. Clearly, the definition of projectively flat spacetime (2) implies the inertial motion. If x^0 denotes time, one could obtain the formula (1) from (2). In Riemannian manifold, Beltrami's theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. It is well known that there are three kinds of spacetime with constant sectional curvature. They are Minkowski (Mink) spacetime and dS/AdS spacetime. That is why there only exists three classes of inertial frames in Riemannian spacetime. The three classes of inertial frames are the basis of the dSSR.

If we further loose the requirement for spacetime, just require that the spacetime should be Finslerian, various inertial frames could be obtained, including the inertial frames for VSR and DSR.

Instead of defining an inner product structure over the tangent bundle in Riemann geometry, Finsler geometry is based on the so called Finsler structure F with the property $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$, where $x \in M$ represents position and $y \equiv \frac{dx}{d\tau}$ represents velocity. The Finsler metric is given as[20]

$$g_{\mu\nu} \equiv \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} \left(\frac{1}{2}F^2\right). \tag{3}$$

Finsler geometry has its genesis in integrals of the form

$$\int_s^r F(x^1, \cdots, x^n; \frac{dx^1}{d\tau}, \cdots, \frac{dx^n}{d\tau})d\tau . \tag{4}$$

The Finsler structure represents the length element of Finsler space. Two types of Finsler space should be noticed. One is the Riemann space. A Finsler metric is said to be Riemannian, if F^2 is quadratic in y. Another is locally Minkowski space. A Finsler metric is said to be locally Minkowskian if at every point, there is a local coordinate system, such that $F = F(y)$ is independent of the position x [20].

The geodesic equation for Finsler manifold is given as[20]

$$\frac{d^2 x^\mu}{d\tau^2} + 2G^\mu = 0, \tag{5}$$

where

$$G^\mu = \frac{1}{4}g^{\mu\nu}\left(\frac{\partial^2 F^2}{\partial x^\lambda \partial y^\nu}y^\lambda - \frac{\partial F^2}{\partial x^\nu}\right) \tag{6}$$

is called geodesic spray coefficient. Obviously, if F is Riemannian metric, then

$$G^\mu = \frac{1}{2}\gamma^\mu_{\nu\lambda}y^\nu y^\lambda, \tag{7}$$

where $\gamma^\mu_{\nu\lambda}$ is the Riemannian Christoffel symbol. By making use of the geodesic equation (5), one could find that a Finsler metric is locally projectively flat if and only if G^μ satisfies

$$G^\mu = P(x,y)y^\mu, \tag{8}$$

where $P(x,y)$ is a function of x and y. It is equivalent to the following equation that was proposed by Hamel [21]

$$\frac{\partial^2 F}{\partial x^\lambda \partial y^\nu}y^\lambda = \frac{\partial F}{\partial x^\nu}. \tag{9}$$

By making use of the Hamel equation (9), we get

$$G^\mu = \left(\frac{\partial F}{\partial x^\nu}y^\nu/2F\right)y^\mu. \tag{10}$$

It means that $P = \frac{\partial F}{\partial x^\nu}y^\nu/2F$. One should notice that

$$\begin{aligned}\frac{dF}{d\tau} &= \frac{\partial F}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \frac{\partial F}{\partial y^\mu}\frac{dy^\mu}{d\tau} \\ &= 2PF - 2PF = 0,\end{aligned} \tag{11}$$

where we has already used the formula for P and the geodesic equation (5) to deduce the second equation.

III. SYMMETRY IN FINSLER SPACE

To investigate the Killing vectors, we should construct the isometric transformations of Finsler structure. It is convenient to discuss the isometric transformations under an infinitesimal coordinate transformation for x

$$\bar{x}^\mu = x^\mu + \epsilon V^\mu, \tag{12}$$

together with a corresponding transformation for y

$$\bar{y}^\mu = y^\mu + \epsilon\frac{\partial V^\mu}{\partial x^\nu}y^\nu, \tag{13}$$

where $|\epsilon| \ll 1$. Under the coordinate transformation (12) and (13), to first order in $|\epsilon|$, we obtain the expansion of the Finsler structure,

$$\bar{F}(\bar{x},\bar{y}) = \bar{F}(x,y) + \epsilon V^\mu\frac{\partial F}{\partial x^\mu} + \epsilon y^\nu\frac{\partial V^\mu}{\partial x^\nu}\frac{\partial F}{\partial y^\mu}, \tag{14}$$

where $\bar{F}(\bar{x},\bar{y})$ should equal to $F(x,y)$. Under the transformation (12) and (13), a Finsler structure is called isometry if and only if

$$F(x,y) = \bar{F}(x,y). \tag{15}$$

Deducing from the (14), we obtain the Killing equation $K_V(F)$ in Finsler space

$$K_V(F) \equiv V^\mu\frac{\partial F}{\partial x^\mu} + y^\nu\frac{\partial V^\mu}{\partial x^\nu}\frac{\partial F}{\partial y^\mu} = 0. \tag{16}$$

Searching the Killing vectors for general Finsler manifold is a difficult task. Here, we give the Killing vectors for a class of Finsler space-(α, β) space[22] with metric defining as

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{17}$$

$$\alpha = \sqrt{a_{\mu\nu}y^\mu y^\nu} \text{ and } \beta = b_\mu(x)y^\mu, \tag{18}$$

where $\phi(s)$ is a smooth function, α is a Riemannian metric and β is a one form. Then, the Killing equation (16) in (α, β) space reads

$$
\begin{aligned}
0 &= K_V(\alpha)\phi(s) + \alpha K_V(\phi(s)) \\
&= \left(\phi(s) - s\frac{\partial\phi(s)}{\partial s}\right) K_V(\alpha) + \frac{\partial\phi(s)}{\partial s}K_V(\beta).
\end{aligned} \tag{19}
$$

And by making use of the Killing equation (16), we obtain

$$K_V(\alpha) = \frac{1}{2\alpha}(V_{\mu|\nu} + V_{\nu|\mu})y^\mu y^\nu, \tag{20}$$

$$K_V(\beta) = \left(V^\mu\frac{\partial b_\nu}{\partial x^\mu} + b_\mu\frac{\partial V^\mu}{\partial x^\nu}\right)y^\nu, \tag{21}$$

where "|" denotes the covariant derivative with respect to the Riemannian metric α. The solutions of the Killing equation (19) have three viable scenarios. The first one is

$$\phi(s) - s\frac{\partial\phi(s)}{\partial s} = 0 \text{ and } K_V(\beta) = 0, \tag{22}$$

which implies $F = \lambda\beta$ for all $\lambda \in \mathbb{R}$. The second one is

$$\frac{\partial\phi(s)}{\partial s} = 0 \text{ and } K_V(\alpha) = 0, \tag{23}$$

which implies $F = \lambda\alpha$ for all $\lambda \in \mathbb{R}$. The above two scenarios are just trivial space. Here we focus on the case of $\phi(s) - s\frac{\partial\phi(s)}{\partial s} \neq 0$ and $\frac{\partial\phi(s)}{\partial s} \neq 0$. This will induce the last scenario.

Apparently, in the last scenario we have such solutions

$$V_{\mu|\nu} + V_{\nu|\mu} = 0, \tag{24}$$

$$V^\mu\frac{\partial b_\nu}{\partial x^\mu} + b_\mu\frac{\partial V^\mu}{\partial x^\nu} = 0. \tag{25}$$

The equation (24) is no other than the Riemannian Killing equation. The equation (25) can be regarded as the constraint on the Killing vectors that satisfy the Killing equation (24). Here, we must point out that additional solutions of Killing equation (19) for (α, β) space exist, besides the solutions (24) and (25). It will be discussed in next section.

However, the Killing equation for one type of (α, β) space-Randers space[23] only have solutions (24) and (25). In Randers space, the $\phi(s)$ is set as $\phi(s) = 1 + s$. Then, the Killing equation (19) reduces to

$$K_V(\alpha) + K_V(\beta) = 0. \tag{26}$$

The $K_V(\alpha)$ contains irrational term of y^μ and $K_V(\beta)$ only contains rational term of y^μ, therefore the equation (26) satisfies if and only if $K_V(\alpha) = 0$ and $K_V(\beta) = 0$.

IV. LIE ALGEBRA AND KINEMATICS IN PROJECTIVELY FLAT (α, β) SPACETIME

An n $(n > 3)$ dimensional (α, β) space is projectively flat with constant flag curvature if and only if one of the following holds[24]

A. it is Riemann spacetime with constant sectional curvature;

B. it is locally Minkowski spacetime;

C. it is locally isometric to a generalized Funk spacetime[25];

D. it is locally isometric to Berwald's metric spacetime[26].

We will discuss the four types of projectively flat space respectively. Throughout this section the $\cdot$ denotes the inner product of Minkowski space $x \cdot x = \eta_{\mu\nu} x^\mu x^\nu$, where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

A. Symmetry in type A (α, β) spacetime and dSSR

The metric of Riemann spacetime with constant sectional curvature can be given by the projective coordinate system

$$F_R = \frac{\sqrt{(y \cdot y)(1 - \mu(x \cdot x)) + \mu(x \cdot y)^2}}{1 - \mu(x \cdot x)}, \tag{27}$$

where the sectional curvature μ of metric (27) is constant. Clearly, the signature $+, 0, -$ of μ corresponds to the dS spacetime, Mink spacetime and AdS spacetime, respectively. Such a metric (27) is invariant under the fractional linear transformations (FLT), and it is $ISO(1,3)/SO(1,4)/SO(2,3)$- invariant Mink/dS/AdS-spacetime[10].

By making use of the formula (6), we know that the geodesic spray coefficient G^μ for metric (27) is given as

$$G_R^\mu = \frac{\mu(x \cdot y)}{1 - \mu(x \cdot x)} y^\mu. \tag{28}$$

Thus, the geodesic equation for metric (27) is of the form

$$\frac{d^2 x^\mu}{d\tau^2} + \frac{2\mu(x \cdot \frac{dx}{d\tau})}{1 - \mu(x \cdot x)} \frac{dx^\mu}{d\tau} = 0. \tag{29}$$

In fact, the geodesic equation is equivalent to

$$\frac{dp^\mu}{d\tau} = 0, \quad p^\mu \equiv \frac{m_R}{F_R} \frac{1}{1 - \mu(x \cdot x)} \frac{dx^\mu}{d\tau}, \tag{30}$$

where m_R is the mass of the particle. Thus, p^μ is a constant along the geodesic. It could be regarded as the counterpart of momentum. From $F_R^2 = g_{\mu\nu} y^\mu y^\nu$, we get

$$g_{\mu\nu} p^\mu p^\nu = \frac{1}{(1 - \mu(x \cdot x))^2} m_R^2. \tag{31}$$

It is obvious that if $\mu = 0$, the above relation returns to the dispersion relation in Minkowski spacetime. The counterpart of angular momentum tensor could be defined as

$$L^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu. \tag{32}$$

It is also a conserved quantities along the geodesic, for $\frac{dL^{\mu\nu}}{d\tau} = 0$. The dispersion law in dSSR [10] is given as

$$p \cdot p - \frac{|\mu|}{2} L \cdot L = m_R^2. \tag{33}$$

By making use of the Killing equation (16), we obtain the Killing vectors for Riemmannian metric (27)

$$V^\mu = Q^\mu_{\ \nu} x^\nu + C^\mu - \mu(x \cdot C) x^\mu, \tag{34}$$

where $Q_{\mu\nu} = \eta_{\rho\mu} Q^\rho_{\ \nu}$ is an arbitrary constant skew-symmetric matrix and $C_\mu = \eta_{\rho\mu} C^\rho$ is an arbitrary constant vector. The isometric group of a Finsler space is a Lie group [18]. One should notice that translation-like generators are induced by C^μ and Lorentz generators are induced by $Q_{\mu\nu}$. The generators of isometric group in Riemannian space (27) read

$$\eta_{\mu\nu} \hat{p}^\nu = \hat{p}_\mu = i(\partial_\mu - \mu x_\mu (x \cdot \partial)), \tag{35}$$

$$\hat{L}_{\mu\nu} = x_\mu \hat{p}_\nu - x_\nu \hat{p}_\mu = i(x_\mu \partial_\nu - x_\nu \partial_\mu). \tag{36}$$

The non-trivial Lie algebra corresponded to the Killing vectors (34) is given as

$$
\begin{aligned}
[\hat{p}_\mu, \hat{p}_\nu] &= i\mu \hat{L}_{\mu\nu}, \\
[\hat{L}_{\mu\nu}, \hat{p}_\rho] &= i\eta_{\nu\rho}\hat{p}_\mu - i\eta_{\mu\rho}\hat{p}_\nu, \\
[\hat{L}_{\mu\nu}, \hat{L}_{\rho\lambda}] &= i\eta_{\mu\lambda}\hat{L}_{\nu\rho} - i\eta_{\mu\rho}\hat{L}_{\nu\lambda} + i\eta_{\nu\rho}\hat{L}_{\mu\lambda} - i\eta_{\nu\lambda}\hat{L}_{\mu\rho}.
\end{aligned}
\tag{37}
$$

While sectional curvature of Riemannian spacetime (27) μ vanishes, the dS/AdS spacetime reduce to Mink spacetime, the momentum tensors and angular momentum tensors reduce to the one in Mink spacetime, and the Lie algebra (37) in dSSR reduces to Poincare algebra.

The sectional curvature μ is linked with [10] the cosmological constant Λ [27], and the Newton-Hooke constant ν [28]

$$
\mu \simeq \frac{\Lambda}{3}, \quad \nu \equiv c\sqrt{\mu} \sim 10^{-35}s^{-2} .
\tag{38}
$$

B.   Symmetry in type B (α, β) spacetime and VSR

A Finsler metric is said to be locally Minkowskian if at every point, there is a local coordinate system, such that $F = F(y)$ is independent of the position x. It is clear from the definition (6) that the geodesic spray coefficient G^μ vanishes in locally Minkowski space. Thus, the geodesic equation of locally Minkowshi space is of simply form

$$
\frac{d^2 x^\mu}{d\tau^2} = 0.
\tag{39}
$$

The momentum tensor $p^\mu = \frac{m_V}{F_V}\frac{dx^\mu}{d\tau}$ and angular momentum tensor $L^{\mu\nu} \equiv x^\mu p^\nu - x^\nu p^\mu$ are conserved quantities along the geodesic, for

$$
\frac{dp^\mu}{d\tau} = 0, \quad \frac{dL^{\mu\nu}}{d\tau} = 0.
\tag{40}
$$

Besides the Minkowski space, locally Minkowski space still involve a various types of metric space. But not all of them has physical implication. Here, we just focus on the locally Minkowski space which is invariant under the VSR symmetric group.

The VSR preserves the law of energy-momentum conservation[6]. It implies that the translation invariance should be contained in the symmetries of the VSR. The left symmetries of the VSR include four possible subgroups of Lorentz group. We introduce the notation $T_1 = (K_x + J_y)/\sqrt{2}$ and $T_2 = (K_y - J_x)/\sqrt{2}$ (the index x, y, z denote the space coordinate), where J and K are the generators of rotations and boosts, respectively. The four subgroups of Lorentz group are given as[11]:

i)$T(2)$, the Abelian subgroup of the Lorentz group, generated by T_1 and T_2, with the structure:

$$
[T_1, T_2] = 0;
\tag{41}
$$

ii)$E(2)$, the group of two-dimensional Euclidean motion, generated by T_1, T_2 and J_z, with the structure:

$$
[T_1, T_2] = 0, \ [J_z, T_1] = -iT_2, \ [J_z, T_2] = iT_1;
\tag{42}
$$

iii)$HOM(2)$, the group of orientation-preserving similarity transformations, generated by T_1, T_2 and K_z, with the structure:

$$
[T_1, T_2] = 0, \ [T_1, K_z] = iT_1, \ [T_2, K_z] = iT_2;
\tag{43}
$$

iv)$SIM(2)$, the group isomorphic to the four-parametric similitude group, generated by T_1, T_2, J_z and K_z, with the structure:

$$
\begin{aligned}
&[T_1, T_2] = 0, \quad [T_1, K_z] = iT_1 \quad, \ [T_2, K_z] = iT_2, \\
&[J_z, K_z] = 0, \quad [J_z, T_1] = -iT_2 \quad, \ [J_z, T_2] = iT_1.
\end{aligned}
\tag{44}
$$

We will show that there is a relation between the isometric group of the Finsler structure[15]

$$
F_V = (\eta_{\mu\nu}y^\mu y^\nu)^{(1-n)/2}(b_\rho y^\rho)^n
\tag{45}
$$

and symmetries of the VSR. Here n is an arbitrary constant, $\eta_{\mu\nu}$ is Minkowskian metric and $b_\rho = \eta_{\mu\rho}b^\mu$ is a constant vector. It is referred as the VSR metric. By making use of the Killing equation (16), we obtain Killing equation for the VSR metric

$$y^\nu \frac{\partial V^\mu}{\partial x^\nu} \left(\frac{(1-n)y_\mu(b_\rho y^\rho)^n + n(\eta_{\alpha\beta}y^\alpha y^\beta)^{1/2}b_\mu(b_\rho y^\rho)^{n-1}}{(\eta_{\alpha\beta}y^\alpha y^\beta)^{(1+n)/2}} \right) = 0. \tag{46}$$

The Eq. (46) has solutions

$$V^\mu = Q^\mu{}_\nu x^\nu + C^\mu, \tag{47}$$
$$b_\mu Q^\mu{}_\nu = 0, \tag{48}$$

where $Q_{\mu\nu} = \eta_{\rho\mu}Q^\rho{}_\nu$ is an arbitrary constant skew-symmetric matrix and $C_\mu = \eta_{\rho\mu}C^\rho$ is an arbitrary constant vector. If one requires that the transformation group for the vectors no other than the Lorentz one or subgroup of Lorentz one, formula (47) togethers with the constraint (48) is the only solution of Killing equation (16) for the VSR metric.

Taking the light cone coordinate [29] $\eta_{\alpha\beta}y^\alpha y^\beta = 2y^+y^- - y^i y^i$ (with i ranging over the values 1 and 2) and supposing $b_\mu = \{0, 0, 0, b_-\}(b_- = 1)$, we know that in general $Q^-{}_\mu \neq 0$. It means that the Killing vectors of the VSR metric (45) do not have non-trivial components Q_{+-} and Q_{+i}. The isometric group of a Finsler space is a Lie group [18]. The non-trivial Lie algebra corresponded to the Killing vectors (47), which satisfies the constraint (48), is given as

$$[J_z, T^i] = i\epsilon_{ij}T^j, \quad [J_z, P^i] = i\epsilon_{ij}P^j,$$
$$[T_i, P^-] = -iP_i, \quad [T_i, P^j] = -i\delta_{ij}P^+, \tag{49}$$

where $\epsilon_{12} = -\epsilon_{12} = 1, \epsilon_{11} = \epsilon_{22} = 0$ and $P^\pm = (P_0 \pm P_z)/\sqrt{2}$. It is obvious that the generators of the isometric group of the VSR metric include generators of $E(2)$ and four spacetime translation generators. This result induces the $E(2)$ scenario of VSR from the VSR metric (45). The $HOM(2)$ scenario of VSR could be induced in the same approach.

The above investigations are under the premise that the direction of spacetime is arbitrary or the transformation group for the vectors no other than the Lorentz group or subgroups of Lorentz group. It means that no preferred direction exists in spacetime. If the spacetime does have a special direction, the Killing equation (16) for the VSR metric will have a special solution. The VSR metric was first suggested by Bogoslovsky [30]. He assumed that the spacetime has a preferred direction. Following the assumption and taking the null direction to be the preferred direction, we obtain the solution of Killing equation (46)

$$V^\mu = (Q^\mu{}_\nu + \delta^\mu{}_\nu)x^\nu + C^\mu, \tag{50}$$

where $Q^\mu{}_\nu$ is an antisymmetrical matrix and satisfies the requirement

$$Q_{+-}n^- = -n^-. \tag{51}$$

Here n^- is a null direction. One can check that the Killing vectors (50) does not have non-trivial components Q_{+i}. It implies that the null direction is invariant under the transformation

$$\Lambda^-{}_- n^- \equiv \left(\delta^-{}_- + \epsilon(n\delta^-{}_- + Q^-{}_-) \right) n^- = (1 + \epsilon(n-1))n^-. \tag{52}$$

Here, $\Lambda^\mu{}_\nu$ denotes the counterpart of Lorentz transformation. Therefore, if the spacetime has a preferred direction in null direction, the symmetry corresponded to Q_{+-} is restored. One can see that the Killing vectors (50) have a non-trivial component $\delta^\mu{}_\nu x^\nu$. It represents the dilations. Thus, we know that the transformation group for the VSR metric (45) contains dilations, while the null direction is a preferred direction. One could obtain the Lie algebra for such transformation group. In fact, the non-trivial Lie algebra is just the algebra of $DISIM(2)$ group proposed by Gibbons et al.[15]

$$[K_z, P^\pm] = -i(n \pm 1)P^\pm, \quad [K_z, P^i] = -inP^i,$$
$$[K_z, T_i] = -iT_i, \quad [J_z, T^i] = i\epsilon_{ij}T^j,$$
$$[J_z, P^i] = i\epsilon_{ij}P^j, \quad [T_i, P^-] = -iP_i,$$
$$[T_i, P^j] = -i\delta_{ij}P^+. \tag{53}$$

The $DISIM(2)$ group is a subgroup of Weyl group, it contains a subgroup $E(2)$ together with a combination of a boost in the $+-$ direction and a dilation. It should be noticed that the deformed generator K_z acts not only as a boost but also a dilation. The transformation acts by K_z is given as

$$\bar{x}^\pm = (\exp(\phi))^{\pm 1+n} x^\pm, \quad \bar{x}^i = (\exp(\phi))^n x^i, \tag{54}$$

where $\exp(\phi) = \sqrt{\frac{1+v/c}{1-v/c}}$. The transformations act by other generators of $DISIM(2)$ group are same with Lorentz one.

If b_μ in the VSR metric (45) has the form $b_\mu = \{0, b_x, 0, b_-\}(b_x = b_- = 1)$, solutions of Killing equation (46) show that the Killing vectors just have non-trivial components Q_{-y} and C^μ. However, the corresponded Lie algebra does not exist. For the generators corresponded to Q_{-y} together with the generators of translations can not form a subalgebra of the Poincare algebra. Consequently, we show that the investigation of Killing equation for VSR metric (45) could account for the $E(2)$, $HOM(2)$ and $SIM(2)(DISIM(2))$ scenarios of the VSR.

The Lagrangian for VSR metric is given as

$$\mathcal{L} = m_V F_V = m_V (\eta_{\mu\nu} y^\mu y^\nu)^{(1-n)/2} (b_\rho y^\rho)^n. \tag{55}$$

The corresponding dispersion relation is of the form

$$\eta^{\mu\nu} p_\mu p_\nu = m_V^2 (1 - n^2) \left(\frac{n^\rho p_\rho}{m(1-n)} \right)^{2n/(1+n)}. \tag{56}$$

The dispersion relation (56) is not Lorentz-invariant, but invariant under the transformations of $DISIM(2)$ group. Ref. [30] showed that the ether-drift experiments gives a constraint $|n| < 10^{-10}$ for the parameter n of the VSR metric (45).

C.   Symmetry in type C (α, β) spacetime

The generalized Funk metric [25] has two geometrical parameters. For physical consideration and simplicity, as DSR, VSR and dSSR, only one geometrical parameter is needed. Therefore, we just investigate the Funk metric of this form

$$F_F = \frac{\sqrt{(y \cdot y)(1 - \kappa^2(x \cdot x)) + \kappa^2(x \cdot y)^2} - \kappa(x \cdot y)}{1 - \kappa^2(x \cdot x)}. \tag{57}$$

Apparently, the Funk metric (57) is of Randers type,

$$F_F = \alpha_F + \beta_F, \quad \alpha_F = \frac{\sqrt{(y \cdot y)(1 - \kappa^2(x \cdot x)) + \kappa^2(x \cdot y)^2}}{1 - \kappa^2(x \cdot x)}, \quad \beta_F = \frac{-\kappa(x \cdot y)}{1 - \kappa^2(x \cdot x)}. \tag{58}$$

As discussed in Sec.3, the Killing vectors of Funk metric of Randers type must satisfy both $K_V(\alpha) = 0$ and $K_V(\beta) = 0$, and it is the only solutions of the Killing equation (16). The solution of equation $K_V(\alpha) = 0$ gives

$$V^\mu = Q^\mu{}_\nu x^\nu + C^\mu - \kappa^2 (x \cdot C) x^\mu, \tag{59}$$

where $Q_{\mu\nu} = \eta_{\rho\mu} Q^\rho{}_\nu$ is an arbitrary constant skew-symmetric matrix and $C_\mu = \eta_{\rho\mu} C^\rho$ is an arbitrary constant vector. And the solution of equation $K_V(\beta) = 0$ gives

$$\kappa C^\nu = 0. \tag{60}$$

The solutions (59) and (60) imply that the Killing vectors of Funk metric (57) is of the form

$$V^\mu = Q^\mu{}_\nu x^\nu, \tag{61}$$

if $\kappa \neq 0$. While $\kappa = 0$, the Funk metric (57) reduces to Minkowski metric, the solutions (59) and (60) reduce to

$$V^\mu = Q^\mu{}_\nu x^\nu + C^\mu, \tag{62}$$

as expected. The non-trivial Lie algebra of non-trivial Funk spacetime (57) $(\kappa \neq 0)$ corresponded to the Killing vectors (61) is given as

$$[\hat{L}_{\mu\nu}, \hat{L}_{\rho\lambda}] = i\eta_{\mu\lambda}\hat{L}_{\nu\rho} - i\eta_{\mu\rho}\hat{L}_{\nu\lambda} + i\eta_{\nu\rho}\hat{L}_{\mu\lambda} - i\eta_{\nu\lambda}\hat{L}_{\mu\rho}, \tag{63}$$

where $\hat{L}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. It means that the non-trivial Funk metric (57) is invariant just under the Lorentz group.

By making use of the formula (6), the geodesic spray coefficient G^μ for metric (57) is given as

$$G_F^\mu = -\kappa \frac{F_F}{2} y^\mu. \tag{64}$$

Thus, the geodesic equation for metric (57) is given as

$$\frac{d^2 x^\mu}{d\tau^2} - \kappa F_F \frac{dx^\mu}{d\tau} = 0. \tag{65}$$

Actually, the geodesic equation (65) is related to the scaled Berwald's metric F_B, which will be discussed in the next subsection. And the geometrical parameter in F_B is set as κ. The derivative of F_B with respect to the curve parameter τ in Funk metric (57) reads

$$\begin{aligned}
\frac{dF_B}{d\tau} &= \frac{\partial F_B}{\partial x^\mu} \frac{dx^\mu}{d\tau} + \frac{\partial F_B}{\partial y^\mu} \frac{dy^\mu}{d\tau} \\
&= -2\kappa F_B F_F + \kappa F_B F_F \\
&= -\kappa F_B F_F.
\end{aligned} \tag{66}$$

Therefore, the geodesic equation (65) is equivalent to

$$\frac{dp^\mu}{d\tau} = 0, \quad p^\mu \equiv \frac{m_F F_B}{F_F^2} \frac{dx^\mu}{d\tau}, \tag{67}$$

where m_F is the mass of the particle in Funk spacetime.

The dispersion relation in Funk spacetime (57) is given as

$$F_F^2(x,p) = g_{\mu\nu} p^\mu p^\nu = m_F^2 \frac{F_F^2(x,y)}{F_B^2(x,y)}. \tag{68}$$

The flag curvature of Funk spacetime is $K_F = \frac{1}{4}\kappa^2$. As the discussion about dSSR, the constant flag curvature may relate to new physical scale (like cosmological constant), and it is very tiny. Therefore, such counterpart of special relativity-Funk special relativity also cannot be excluded by the experiments. To first order in κ, we obtain the expansion of the dispersion relation (68)

$$p \cdot p - 2\kappa(x \cdot p)\sqrt{p \cdot p} = m_F^2. \tag{69}$$

Such dispersion relation (69) could be regarded as one type of modified dispersion law in DSR.

D. Symmetry in type D (α, β) spacetime

The metric constructed by Berwald [26] is of the form

$$F = \frac{\left(\sqrt{(y \cdot y)(1 - x \cdot x) + (x \cdot y)^2} + (x \cdot y)\right)^2}{(1 - (x \cdot x))^2 \sqrt{(y \cdot y)(1 - x \cdot x) + (x \cdot y)^2}}. \tag{70}$$

It is projectively flat with constant flag curvature $K_B = 0$. One important property of projective geometry shows that a projectively flat space is still projectively flat after a scaling on x. It can be proved by using the Hamel equation (9). Thus, the scaled Berwald's metric is given as

$$F_B = \frac{\left(\sqrt{(y \cdot y)(1 - \lambda^2 (x \cdot x)) + \lambda^2 (x \cdot y)^2} - \lambda(x \cdot y)\right)^2}{(1 - \lambda^2 (x \cdot x))^2 \sqrt{(y \cdot y)(1 - \lambda^2 (x \cdot x)) + \lambda^2 (x \cdot y)^2}}, \tag{71}$$

where λ is a constant. The flag curvature of scaled Berwald's metric (71) is $K_B = 0$.

Defining

$$\alpha_B = \frac{\sqrt{(y \cdot y)(1 - \lambda^2 (x \cdot x)) + \lambda^2 (x \cdot y)^2}}{1 - \lambda^2 (x \cdot x)}, \quad \beta_B = \frac{-\lambda(x \cdot y)}{1 - \lambda^2 (x \cdot x)}, \tag{72}$$

we have

$$F_B = \frac{(\alpha_B + \alpha_B)^2}{\alpha_B(1 - x \cdot x)}. \tag{73}$$

Substituting the metric (73) into the Killing equation (16), we get

$$K_V(F_B) = \frac{\alpha_B + \beta_B}{\alpha_B(1 - \lambda^2(x \cdot x))} \left((1 - \beta_B/\alpha_B)K_V(\alpha_B) + 2K_V(\beta_B) + 2\lambda^2(\alpha_B + \beta_B)\frac{x_\mu V^\mu}{1 - \lambda^2(x \cdot x)} \right) = 0. \tag{74}$$

The equations $K_V(\alpha_B) = 0$ and $K_V(\beta_B) = 0$ imply

$$V^\mu = Q^\mu{}_\nu x^\nu, \tag{75}$$

if $\lambda \neq 0$, where $Q_{\mu\nu} = \eta_{\rho\mu}Q^\rho{}_\nu$ is an arbitrary constant skew-symmetric matrix. Furthermore, it is obvious that $x_\mu Q^\mu{}_\nu x^\nu = 0$. Therefore, the Killing vectors of the form (75) is a solution of the Killing equation (74). The Killing vector of the form (75) means that the scaled Berwald's metric spacetime (71) is isotropic about a given point. Therefore, the Killing vectors which implies such symmetry (isotropic about a given point) reach its maximal numbers. And additional solutions of Killing equations (74) must have the form

$$V^\mu = f^\mu(x, C), \tag{76}$$

where $C_\mu = \eta_{\rho\mu}C^\rho$ is an arbitrary constant vector. If $V^\mu = \{f(x, c), 0, 0, 0\}$ is a solution of Killing equation (74), it is clear that $V^\mu = \{f(x, c), f(x, c), f(x, c), f(x, c)\}$ is also a solution of (74). Therefore, the maximal dimension of isometric group of 4 dimensional scaled Berwald's spacetime equals either 6 or 10. It is known [31] that the maximal dimension of isometric group in an n dimensional non Riemannian Finslerian space is $\frac{n(n-1)}{2} + 2$. The scaled Berwald's metric spacetime is non Riemannian. We conclude that the solution of Killing equation (74) only have solutions of the form (75).

The Lie algebra of non-trivial scaled Berwald's metric spacetime (71) ($\lambda \neq 0$) corresponded to the Killing vectors (75) is given as

$$[\hat{L}_{\mu\nu}, \hat{L}_{\rho\lambda}] = i\eta_{\mu\lambda}\hat{L}_{\nu\rho} - i\eta_{\mu\rho}\hat{L}_{\nu\lambda} + i\eta_{\nu\rho}\hat{L}_{\mu\lambda} - i\eta_{\nu\lambda}\hat{L}_{\mu\rho}, \tag{77}$$

where $\hat{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$.

By making use of the formula (6), we obtain the geodesic spray coefficient G^μ for metric (71)

$$G_B^\mu = -\lambda\frac{\sqrt{(y \cdot y)(1 - \lambda^2(x \cdot x)) + \lambda^2(x \cdot y)^2} - \lambda(x \cdot y)}{1 - \lambda^2(x \cdot x)}y^\mu = -\lambda F_F y^\mu, \tag{78}$$

where F_F is the Funk metric, and the parameter in F_F is set as λ. Thus, the geodesic equation for metric (71) is given as

$$\frac{d^2x^\mu}{d\tau^2} - 2\lambda F_F\frac{dx^\mu}{d\tau} = 0. \tag{79}$$

One should notice that the derivatives of F_F with respect to the curve parameter τ in scaled Berwald's metric (71) reads

$$\frac{dF_F}{d\tau} = \frac{\partial F_F}{\partial x^\mu}\frac{dx^\mu}{d\tau} + \frac{\partial F_F}{\partial y^\mu}\frac{dy^\mu}{d\tau} = -\lambda F_F^2 + 2\lambda F_F^2 = -\lambda F_F^2. \tag{80}$$

Therefore, the geodesic equation (79) is equivalent to

$$\frac{dp^\mu}{d\tau} = 0, \quad p^\mu \equiv \frac{m_B F_F^2}{F_B^3}\frac{dx^\mu}{d\tau}, \tag{81}$$

where m_B is the mass of the particle in scaled Berwald's metric spacetime. The dispersion relation in scaled Berwald's metric spacetime (71) is given as

$$F_B^2(x, p) = g_{\mu\nu}p^\mu p^\nu = m_B^2\frac{F_F^4(x, y)}{F_B^4(x, y)}. \tag{82}$$

The parameter λ in scaled Berwald's metric spacetime (71) may relate to new physical scale and it is very tiny. To first order in λ, we obtain the expansion of the dispersion law (82)

$$p \cdot p - 4\lambda(x \cdot p)\sqrt{p \cdot p} = m_B^2. \tag{83}$$

Here, we find that Funk spacetime (57) and scaled Berwald's metric spacetime (71) have same isometric group. And the geodesic equations in Funk spacetime and scaled Berwald's metric spacetime are alike, if they both take the same geometrical parameter. Also, to first order in geometrical parameter, the dispersion relation are almost the same.

V. CONCLUSION

In this paper, we have extended the concept of inertial motion in the framework of the projective geometry. The inertial frames in projectively flat Finsler spacetime are investigated. We have studied the inertial motion in a special Finsler spacetime-the projectively flat (α, β) spacetime with constant flag curvature (the counterpart of sectional curvature). The projectively flat (α, β) spacetime with constant flag curvature can be divided into four types. We have showed that the inertial motion and symmetry in Type A and Type B spacetime are just the one in dSSR and VSR, respectively. And the dispersion law in Type C and Type D could be regarded as one types of modified dispersion law in DSR. The four types of (α, β) spacetime involve two parameters-the speed of light and a geometrical parameter which may relate to new physical scale. While the geometrical parameter vanishes, the four types of spacetime reduce to Minkowski spacetime, the momentum tensors and angular momentum tensors reduce to the one in Minkowski spacetime, the corresponded Lie algebra reduces to Poincare algebra, and the inertial motions reduce to the one in special relativity. In the following table, we list basic features of the kinematics and symmetry in the four types spacetime.

TABLE I: the projectively flat (α, β) space with constant flag curvature

Type	parameter	geodesic equation	momentum	isometric group
A	μ	$\frac{d^2 x^\mu}{d\tau^2} + \frac{2\mu(x \cdot \frac{dx}{d\tau})}{1 - \mu(x \cdot x)} \frac{dx^\mu}{d\tau} = 0$	$p_R^\mu \equiv m_R \frac{1}{F_R} \frac{1}{1 - \mu(x \cdot x)} \frac{dx^\mu}{d\tau}$	dS/AdS group
B	n	$\frac{d^2 x^\mu}{d\tau^2} = 0$	$p_V^\mu \equiv m_V \frac{1}{F_V} \frac{dx^\mu}{d\tau}$	DISIM(2) group
C	κ	$\frac{d^2 x^\mu}{d\tau^2} - \kappa F_F(\kappa) \frac{dx^\mu}{d\tau} = 0$	$p_F^\mu \equiv m_F \frac{F_B(\kappa)}{F_F^2(\kappa)} \frac{dx^\mu}{d\tau}$	Lorentz group
D	λ	$\frac{d^2 x^\mu}{d\tau^2} - 2\lambda F_F(\lambda) \frac{dx^\mu}{d\tau} = 0$	$p_B^\mu \equiv m_B \frac{F_F^2(\lambda)}{F_B^3(\lambda)} \frac{dx^\mu}{d\tau}$	Lorentz group

Appendix: Flag curvature

The flag curvature [20, 32] in Finsler geometry is the counterpart of the sectional curvature in Riemannian geometry. It is a geometrical invariant. Furthermore, the same flag curvature is obtained for any connection chosen in Finsler space. The curvature tensor $R^\mu_{\ \nu}$ is defined as

$$R^\mu_{\ \nu}(x, y) \equiv - \left(\frac{\partial G^\mu}{\partial x^\nu} - y^\lambda \frac{\partial^2 G^\mu}{\partial x^\lambda \partial y^\nu} + 2G^\lambda \frac{\partial^2 G^\mu}{\partial y^\lambda \partial y^\nu} - \frac{\partial G^\mu}{\partial x^\lambda} \frac{\partial G^\lambda}{\partial x^\nu} \right), \tag{A.1}$$

where G^μ is geodesic spray coefficient. For a tangent plane $\Pi \subset T_x M$ and a non-zero vector $y \in T_x M$, the flag curvature is defined as

$$K(\Pi, y) \equiv \frac{g_{\lambda\mu} R^\mu_{\ \nu} u^\nu u^\lambda}{F^2 g_{\rho\theta} u^\rho u^\theta - (g_{\sigma\kappa} y^\sigma u^\kappa)^2}, \tag{A.2}$$

where $u \in \Pi$. If F is projectively flat, substituting $G^\mu = P(x, y) y^\mu$ into the definition of flag curvature (A.2), and by making use of formula (A.1), we obtain that

$$K = - \frac{P^2 - \frac{\partial P}{\partial x^\mu} y^\mu}{F^2}. \tag{A.3}$$

The curvature tensor $R^\mu_{\ \nu}$ defined above is presented as $-\bar{R}^\mu_{\ \nu}$ in Ref.[32]. The notation we used here keeps the sectional curvature of dS spacetime to be positive and of AdS spacetime to be negative. By making use of the formula for the flag curvature of projectively flat Finsler spacetime (A.3), we get the flag curvature for dS/AdS spacetime (27), Funk spacetime (57) and scaled Berwald's metric spacetime (71), respectively,

$$K_R = \mu, \quad K_F = \frac{1}{4}\kappa^2, \quad K_B = 0 \tag{A.4}$$

And the flag curvature of locally Minkowski spacetime equals zero.

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